Fig.5 shows the form of the oscillations of the beam at $M = M_*$, i.e. at the boundary of the domain of stability. The values $\omega t = 0, \pi/4, \pi/2, 3\pi/4$ correspond to the curves *1—4*. A wave appears near the mass *M*, moving in a direction opposite to thatof the mass. However, the directions of the motions of the mass nad the wave with respect to the beam are the same.

We note that the system in question can be used as a model of a pipe with a flow of fluid, made thicker at some place (increased mass), in the case when the ratio of the running mass of the pipe and the fluid is small. If the ratio is not small, then additional terms must be introduced in (1) /3/.

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Translated by L.K.

PMM U.S.S.R.,Vol.49,No.4,pp.537-540,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

AVERAGED DESCRIPTION OF THE OSCILLATIONS IN A ONE-DIMENSIONAL, RANDOMLY INHOMOGENEOUS MEDIUM*

A.YU. BELYAEV

The Cauchy problem for a wave equation with coefficients depending randomly on the spatial coordinate is considered. An equation describing the evolution of the expectation of the solution is derived assuming that the fluctuations of the coefficients and the correlation radius are small. The averaged equation, unlike the initial equation, is irreversible with respect to time, and has the form of a one-dimensional equation of motion of a viscoelastic material. The coefficient of effective viscosity obtained is found to be proportional to the intensity of fluctuations of the random characteristics of the inhomogeneous medium.

Numerous problems of the propagation of elastic, electromagentic and other waves in an inhomogeneous medium, reduce to solving the equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[a(x)\frac{\partial u}{\partial x}\right] \tag{1}$$

with initial data for $t = 0$. If the functions $\rho(x)$ and $a(x)$ characterizing the properties of the medium oscillate rapidly, then the problem arises of producing an averaged description of the wave propagation process. In randomly inhomogeneous continua the non-coherent character of wave dispersion by inhomogeneities of the medium produces a decay of solutions, which leads to the irreversibility of the averaged equations.

**Prikl.Matem.Mekhan.*,49,4,696-700,1985

A semi-empirical method of deriving the irreversible averaged equations was used in /1/ in the case of a two-dimensional continuum representing a homogeneous elastic material with randomly distributed inclusions at low concentration. Equations were derived in /2/ for the case when the fluctuations in the characteristics of the medium were small, describing the evolution of the expectation of the solution of (1), by expanding over the corresponding small parameter. It will, however, be shown below that the Cauchy problem with initial data is ill-posed for the equations obtained in it.

In the present paper the equation for determining the averaged solution of the initial problem (1) is constructed using the variational method. The equation has the form of a one-dimensional equation of motion of a homogeneous, viscoelastic material, and the Cauchy problem for it is correct. In the variational approach the problem of initial data is solved automatically, while in /2/ it was not discussed at all.

Let the initial data for (1) have the form

$$u(0, x) = f(x), \quad u_t(0, x) = g(x) \tag{2}$$

Equation (1) can be obtained from the varitional principle /3, 4/

$$\delta H(u) = 0 \tag{3}$$

where the functional $H(u)$ is defined on the set of functions $u(t, x)$ satisfying the first condition of (2) as follows:

$$H(u) = \frac{1}{2}\int_0^T dt \int_{-\infty}^{\infty} dx\,[\rho(x)\,u_t(t, x)\,u_t(T-t, x) + a(x)\,u_x(T-t, x)\,u_x(t, x)] - \int_{-\infty}^{\infty} dx g(x)\,\rho(x)\,u(T, x) \tag{4}$$

The subscripts t and x accompanying the function u denote partial derivatives in t and x, respectively.

The variational principle (3) yields Eq.(1) and the second initial condition of (2).

If the functions $\rho(x)$ and $a(x)$ are random, then the solution of (1) with initial conditions (2) will also be a random function. We shall denote its expectation $M[u(t, x)]$ by $v(t, x)$.

We shall derive the equations for determining $v(t, x)$ using a method described in /5/. The method is as follows. We shall consider the functional $I(u) = M[H(u)]$ on the set of random functions $u(t, x)$ satisfying the first condition of (2). The conditions of stationarity of this functional coincide with the initial equation (1) and second initial condition of (2). We shall vary the functional $I(u)$ in two stages. First we shall find its stationary point under the constraint

$$M[u(t, x)] = v(t, x) \tag{5}$$

where $v(t, x)$ is a non-random function satisfying the condition

$$v(0, x) = f(x) \tag{6}$$

The value of the functional $I(u)$ at the stationary point will represent a functional dependent on v. Let us denote it by $I_0(v)$. The required equation for determining $v(t, x)$ is obtained from the variational principle $\delta I_0(v) = 0$.

The variation of the functional $I(u)$ has the form

$$\delta I(u) = M\left\{\int_0^T dt \int_{-\infty}^{\infty} dx\delta u(T-t, x)\left[\rho(x)\,u_{tt}(t, x) - \frac{\partial}{\partial x}(a(x)\,u_x(t, x))\right] + \int_{-\infty}^{\infty} dx\rho(x)\,[u_t(0, x) - g(x)]\,\delta u(T, x)\right\}$$

By virtue of the first condition of (2), the variations δu vanish at $t = 0$, and have zero expectation by virtue of the constraint (5), otherwise they are arbitrary.

The necessary conditions of stationarity of the functional $I(u)$ have the form

$$\rho(x)\,u_{tt} - (a(x)\,u_x)_x = h(x), \quad \rho(x)\,[u_t(0, x) - g(x)] = p(x) \tag{7}$$

where $h(t, x)$ and $p(x)$ are non-random functions playing the part of Lagrange mutlipliers under the constraint (5). Having solved these equations for $u(t, x)$ and substituted the value obtained into the functional $I(u)$, we obtain the functional $I_0(v)$.

Let us make the following assumption concerning the functions $\rho(x)$ and $a(x)$. Let their values differ only slightly from the expectations, which are assumed independent of x and are denoted by ρ_0 and a_0 respectively. We denote the deviations of the functions $\rho(x)$ and $a(x)$ from ρ_0 and a_0 by $\rho_1(x)$ and $a_1(x)$, and the maximum deviation by δ.

We shall solve Eqs.(7) by expanding the unknown functions $u(t, x), h(t, x), p(x)$ in a power series in terms of the parameter δ. The expressions for the functional $I_0(v)$ calculated with an accuracy of up to and including terms of order δ^2, can be reduced to the form

$$I_0(v) = M\left\{\frac{1}{2}\int_0^T dt \int_{-\infty}^{\infty} dx\,[\rho_0 v_t(T-t, x)\,v_t(t, x) + a_0 v_x(T-t, x)\,v_x(t, x) + \rho_1(x)\,v_t(t, x)\,u_{1t}(T-t, x) + a_1(x)\,v_x(t, x)\,u_{1x}(T-t, x)] - \int_{-\infty}^{\infty} dx\left[\rho_0 g(x)\,v(T, x) + \frac{1}{2}\rho_1(x)\,g(x)\,u_1(T, x)\right]\right\} \tag{8}$$

where the function $u_1(t,x)$ is expressed in terms of $v(t,x)$ as follows:

$$u_1(t,x) = \frac{c}{2a_0}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} dy\,[(a_1(y)\,v_x(\tau,y))_y - \rho_1(y)\,v_{tt}(\tau,y)] -$$
$$\frac{c}{2a_0}\int_{x-ct}^{x+ct} dy\,\rho_1(x)\,[v_t(0,y) - g(y)], \quad c = \left(\frac{a_0}{\rho_0}\right)^{1/2}$$

Further simplifications of expression (8) are based on the assumption that $a_1(x)$ and $\rho_1(x)$ are homogeneous, isotropic random functions with correlation radius much less than the characteristic scale of variation of the functions $f(x)$ and $g(x)$.

Let us introduce the notation

$$\varepsilon_\rho = \rho_0^{-2}\int_{-\infty}^{\infty} M[\rho_1(x)\rho_1(x+y)]\,dy, \quad \varepsilon_a = a_0^{-2}\int_{-\infty}^{\infty} M[a_1(x)\,a_1(x+y)]\,dy$$

The numbers ε_ρ and ε_a are independent of x by virtue of the homogeneity of the random functions $\rho_1(x)$ and $a_1(x)$. It can be shown that they are non-negative and proportional to the correlation radii of these functions. We shall turn our attention to the asymptotic form of $v(t,x)$ when $\varepsilon_a, \varepsilon_\rho$ tends to zero.

If we retain in the functional (8) the principal and first-order terms in the parameters $\varepsilon_a, \varepsilon_\rho$, assuming that the function $v(t,x)$ is independent of $\varepsilon_a, \varepsilon_\rho$, then the functional will take the form

$$I_0(v) = \frac{1}{2}\int_0^T dt \int_{-\infty}^{\infty} dx\,[\rho_0 v_t(T-t,x)\,v_t(t,x) +$$
$$\bar{a} v_x(T-t,x)\,v_x(t,x) - (2c)^{-1}\rho_0\varepsilon_\rho v_t(T-t,x)\,v_{tt}(t,x) +$$
$$(2c)^{-1} a_0\varepsilon_a v_x(T-t,x)\,v_{xt}(t,x)] - \int_{-\infty}^{\infty} dx\,\{\rho_0 g(x)\,v(T,x) +$$
$$(4c)^{-1}\rho_0\varepsilon_\rho[v_t(T,x)\,v_t(0,x) - 2v_t(T,x)\,g(x) + g^2(x)] +$$
$$(4c)^{-1} a_0\varepsilon_a f_{xx}(x)\,v(T,x)\}, \quad \bar{a} = a_0 - a_0^{-1}M[a_1^2(x)]$$

Varying this expression over $v(t,x)$ under the constraint (6) and equating the variation to zero, we obtain

$$\rho_0 v_{tt} - \bar{a} v_{xx} - (2c)^{-1}\rho_0\varepsilon_\rho v_{ttt} - (2c)^{-1} a_0\varepsilon_a v_{xxt} = 0 \tag{9}$$

and some initial conditions for it. We can reduce the system of equations obtained in /2/ to this form. We can show however, that the Cauchy problem is ill-posed for the resulting equation if $\varepsilon_\rho \neq 0$. This follows from the fact that solutions of (9) of the form $e^{i(kx-\omega t)}$ with k real, increasing exponentially with time, exist.

The reason for the ill-posed response is, that in the functional $I_0(v)$ the parameters tending to zero accompany the terms with higher-order derivatives. This leads to the appearance of boundary value problem in x. Equation (9) can, at the same time, by asymptotically well-posed in the sene that the required function $v(t,x)$ satisfies this equation outside the boundary layer with an accuracy of up to terms of order $o(\varepsilon_a + \varepsilon_\rho)$. We cannot, however, use it to obtain the function v.

In such situations, the asymptotic analysis cannot be reduced to expanding the functional in a power series in small parameters. We shall apply the variaional-asymptotic method /6/.

At the first stage we retain in the functional (8) the terms that are principal with respect to the parameters $\varepsilon_a, \varepsilon_\rho$. Varying such a functional under the constraint (6), we obtain the following equation and initial condition:

$$\rho_0 v_{tt} = \bar{a} v_{xx}, \; v_t(0,x) = g(x) \tag{10}$$

We denote the solution of this equation by $v_0(t,x)$, and write the required solution in the form

$$v(t,x) = v_0(t,x) + v_1(t,x)$$

By virtue of the constraint (6) the function $v_1(t,x)$ must vanish at $t=0$. Assuming that v_1 is asymptotically smaller than v_0, we retain in the functional (8) the principal terms in v_1 and principal terms containing both v_0 and v_1. This yields the functional $I_1(v_1)$, which can be written, taking Eq. (10) into account, in the form

$$I_1(v_1) = \frac{1}{2}\int_0^T dt \int_{-\infty}^{\infty} dx\,[\rho_0 v_{1t}(T-t,x)\,v_{1t}(t,x) + \bar{a} v_{1x}(T-t,x)\,v_{1x}(t,x) -$$
$$-\frac{\rho_0\varepsilon_\rho}{c} v_{1t}(T-t,x)\,v_{0tt}(t,x) + \frac{a_0\varepsilon_a}{c} v_{1x}(T-t,x)\,v_{0xt}(t,x)\Big] - \frac{1}{2}\int_{-\infty}^{\infty} dx\,\frac{a_0\varepsilon_a}{c} f_{xx}(x)\,v_1(T,x)$$

Equating the variation of the functional $I_1(v_1)$ to zero, we obtain the equation and initial condition for determining $v_1(t, x)$

$$\rho_0 v_{1tt} - \bar{a} v_{1xx} = (2c)^{-1} (\rho_0 \varepsilon_\rho v_{0ttt} + a_0 \varepsilon_a v_{0xxt}) \tag{11}$$
$$v_{1t}(0, x) = {}^1/_2 c (\varepsilon_\rho + \varepsilon_a) f_{xx}(x)$$

to which we should add the initial condition $v_1(0, x) = 0$ following from the constraint (6). In addition to producing the function $v_0(t, x)$ (10), Eq.(11) yields the solution of the problem in question. We note that the approach adopted here does not give rise to ill-posed problems.

Relations (10) and (11) can be combined within the limits of accuracy used, into a single equation in terms of the function $v(x, t)$ sought

$$\rho_0 v_{tt} - \bar{a} v_{xx} - (2c)^{-1} a_0 (\varepsilon_\rho + \varepsilon_a) v_{xxt} = 0, \quad v(0, x) = f(x),$$
$$v_t(0, x) = g(x) + {}^1/_2 c (\varepsilon_a + \varepsilon_\rho) f_{xx}(x)$$

It has the form of an equation of motion of a one-dimensional viscoelastic medium. Its solution, with the above initial conditions, yields an asymptotically exact value for the averaged solution of, the initial equation (1) when $t \gg c^{-1}(\varepsilon_a + \varepsilon_\rho)$.

When the values of time t are nearly zero, the averaged solution has been shown to have the character of a boundary layer, and more complicated equations are needed for its determination, obtained by varying the functional (8). This explains the appearance of the last term in the second initial condition, which is not present in the exact formulation by virtue of relation (2) and of the definition of the averaged solution. The term in question describes the effect of the temporary boundary layer on the behaviour of the solution at finite times.

The author thanks V.L. Berdichevskii for his interest and for discussing the results.

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Translated by L.K.

PMM U.S.S.R.,Vol.49,No.4,pp.540-544,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
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THE BUBNOV-GALERKIN METHOD IN THE NON-LINEAR THEORY OF HOLLOW, FLEXIBLE MULTILAYER ORTHOTROPIC SHELLS*

V.F. KIRICHENKO, V.A. KRYS'KO and N.S. SUROVA

The existence of solutions of a strongly non-linear system of differential equations describing, in the framework of the kinematic Timoshenko model /1/ adopted for the whole packet in toto /2/, the behaviour of a flexible, multilayer shell whose very layer is made of an inhomogeneous orthotropic material, is proved. To obtain an approximate solution of the problem in question, a procedure is proposed and justified, using the Bubnov-Galerkin (BG) method based on constructing an auxiliary quasilinear system of equations. A similar approach makes it possible to extend the method /3—6/ of studying the convergence of the BG method to strongly non-linear systems of elliptic type equations, and to achieve the convergence of the sequence of approximate solutions to the exact solution in a space of any prescribed degree of smoothness, without imposing additional constraints on the initial data of the problem.

**Prikl.Matem.Mekhan.*,49,4,700-704,1985